Lecture 21

r-Combination with repetition

- Suppose I went to the rise-n-roll that Sophia likes
  - There are 5 flavors
  - I want to buy 8 donuts
  - Infinite supply
  - How many different selections can I make?

$$* \mid * * * \mid * * * \mid * \mid$$

1 2 3 4 5

Code the situation using binary string:

8 zeros (stars)
4 ones (# flavors − 1)
12 bit string

Now how many unique 12 bit strings can be made with 8 zeros and 4 ones

$$\binom{12}{4}$$

More General:

n flavors , r donuts

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

$\boxed{\text{Ex}}$ There are infinite supply of pennies, nickels, dimes, quarters.

- How many ways to select 25 coins?

  25 coins, 4 "flavors"

  $$\binom{25+4-1}{4-1}$$

- How many ways are there to select 25 coins, if at least 5 of them must be quarters?
  - Since at least 5 must be quarters.

  ↳ Ignore these and drop n to 20, since there is only <u>one</u> way to select 5 quarters.

  ↳ so $(1)\cdot\binom{20+4-1}{4-1}$

- How many ways are there to select 25 coins if at most 10 of them are quarters?

  ↳ $x \le 10 \equiv \neg(x \ge 11)$

  ↳ equivalent to $\neg$(at least 11 quarters)

- So # at least 11 quarters

  $$(1)\cdot\binom{14+4-1}{4-1}$$

- So apply counting by complement

$$\underbrace{\binom{25+4-1}{4-1}}_{\substack{\#\text{ of ways to}\\ \text{select 25 coins}}} - \underbrace{\left[(1)\cdot\binom{14+4-1}{4-1}\right]}_{\substack{\#\text{ of ways that}\\ \text{have at least 11 coins}}} = \text{Answer to the problem}$$

Balls in Bins

- How many solutions does the equation $x_1 + x_2 + x_3 = 10$ have if x_1, x_2, x_3 are all non-negative integers?

$$\binom{10+3-1}{3-1}$$

- How many ways are thee to place 10 indistinguishable balls into 3 distinguishable bins?

$$\binom{10+3-1}{3-1}$$

- How many ways to select 10 donuts if there are 3 flavors?

$$\binom{10+3-1}{3-1}$$

- How many solutions does the equation $x_1 + x_2 + x_3 = 10$ have if x_1, x_2, x_3 are all <u>positive</u> integers?

↳ reform question into the one above

$$\left.\begin{array}{ll} x_1 - 1 = y_1 & x_1 = y_1 + 1 \\ x_2 - 1 = y_2 & x_2 = y_2 + 1 \\ x_3 - 1 = y_3 & x_3 = y_3 + 1 \end{array}\right\} \rightarrow x_1 + x_2 + x_3 = y_1 + y_2 + y_3 + 3 = 10$$

$$\rightarrow y_1 + y_2 + y_3 = 7$$

$$\binom{10+3-1}{3-1}$$

So $\binom{7+3-1}{3-1}$

• How many solutions $x_1 + x_2 + x_3 \leq 10$ such that $x_1 + x_2 + x_3$ are non-negative integers?

→ $x_1 + x_2 + x_3 + y = 10$

↳ Now we reformed the problem to have 4 variables with an equality to 10.